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Scale-covariant field theories: II. The pseudo-perturbation expansion

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Abstract. We examine the possibility of developing a perturbation expansion about the pseudo-free scalar theory. The degeneracy of the subtracted scale-covariant equations requires that additional information be provided. We suggest two ways in which this can be done.

1. Introduction

A definition of non-renormalisability that is independent of computational procedure has been proposed by Klauder (1979b, and references therein). The basic idea is that of *discontinuous perturbations*, a common occurrence in quantum mechanics (Klauder 1973, 1979b).

A theory with coupling strength $\lambda \geqslant 0$ is *perturbatively* discontinuous if, on taking $\lambda \to 0^+$, the resulting theory is different from the free theory for which $\lambda = 0$ identically. When this happens the theory given by the $\lambda \to 0^+$ limit is called the *pseudo-free* theory. It is assumed that, when λ is non-zero, a perturbation series in λ can be developed about the pseudo-free theory†.

This has been disputed by Nouri-Moghadam and Yoshimura (1978), who have argued that the branching equations for Green functions in such theories (the analogue of the Schwinger–Dyson equations) do not permit the development of such a 'pseudo-perturbation' expansion.

The purpose of this paper is to re-appraise the arguments of Nouri-Moghadam and Yoshimura (1978) on this and related topics.

As a first step we must recapitulate some aspects of discontinuously perturbative theories. More details are given in the preceding paper of this series (Ebbutt and Rivers 1982a, to be referred to as I).

It has been argued (Klauder 1979a, b, 1981a, b) that discontinuously perturbative theories are accommodated in the path integral formalism by preparing the measure to be scale covariant. That is, a single scalar field φ with $\lambda\varphi^4$ interaction has generating functional (in d Euclidean space–time dimensions)

$$Z'[h] = \int \mathscr{D}'[\varphi] \exp - \int \mathrm{d}x(-\mathrm{i}h\varphi + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m_0^2\varphi^2 + \lambda_0\varphi^4) \qquad (1.1)$$

† It is attempts to develop perturbation series in λ about the *free* theory that give rise to the conventional difficulties of non-renormalisable theories.

($dx \equiv d^d x$) where

$$\mathscr{D}'[\Lambda\varphi] = F[\Lambda]\mathscr{D}'[\varphi] \qquad \Lambda(x) > 0, \forall x \tag{1.2}$$

is a *scale-covariant* measure whose normalisation is chosen so that $Z'[0] = 1$. This contrasts to the translationally invariant measures of canonical renormalisable theories.

On switching off the interaction in (1.1) we recover the *pseudo-free theory*.

Inevitably, formal path integrals like the above cannot be evaluated directly. Rather in the first instance, we attempt to solve the Green function branching equations that follow from

$$\left\{h(x)\frac{\delta}{\delta h(x)} + :\frac{\delta}{\delta h(x)} K_x \frac{\delta}{\delta h(x)}: - 4\lambda_0 :\frac{\delta^4}{\delta h(x)^4}:\right\} Z'[h] = 0 \tag{1.3}$$

where $K_x = -\nabla_x + m_0^2$, satisfied by (1.1).

The : : denotes the subtraction procedure

$$:\frac{\delta^p}{\delta h(x)^p}: Z'[h] = \frac{\delta^p}{\delta h(x)^p} Z'[h] - \frac{\delta^p}{\delta h(x)^p} Z'[h]\bigg|_{h=0} Z'[h]. \tag{1.4}$$

The presence of subtraction in (1.3) follows from the form of (1.2) and is reinforced by the exact results of the independent-value model (IVM) in which K is replaced by m_0^2 (Klauder 1975, 1979a).

As a further indication of the need to be extremely cautious in handling path integrals like (1.1)†, it can be seen (Klauder 1979b) that the definition of operator products implied by the scale-invariant measure $\mathscr{D}'[\varphi]$ implies formally *linear* branching equations for the *connected* Green functions

$$i^n W_n(x_1 x_2 \ldots x_n) = \frac{\delta^n}{\delta h(x_1) \ldots \delta h(x_n)} \ln Z'[h]\bigg|_{h=0}. \tag{1.5}$$

These are

$$\left(\sum_{r=1}^{2m} \delta(x - x_r)\right) W_{2m}(x_1 x_2 \ldots x_{2m}) - \lim_{x' \to x} K_x W_{2m+2}(x, x', x_1 \ldots x_{2m})$$
$$- 4\lambda_0 W_{2m+4}(xxxxx_1 \ldots x_{2m}) = 0 \qquad m \geq 1 \tag{1.6}$$

assuming all W_{2m+1} to be identically zero.

Equations (1.6) have, so far (Klauder 1979a, b), been considered the essential content of the formal expression (1.1). Unfortunately, with the exception of the IVM, and an ingenious solution when $m_0 = \lambda_0 = 0$ (Klauder 1979b), they have proved very resistant to solution.

In an analysis of (1.6) Nouri-Moghadam and Yoshimura (1978) have argued that (i) the equations are extremely degenerate; (ii) in particular, they do *not* permit a perturbation series expansion in λ_0, negating the whole approach of discontinuous perturbations.

To understand the accuracy of these objections, we shall firstly contrast them (in § 2) to the Schwinger–Dyson (SD) equations of the canonical theory.

Secondly, on the assumption that a large part of the problem is purely combinatoric we can examine this aspect in detail for the IVM, contrasting this to the translation-covariant static ultra-local model (SULM) proposed by Caianiello and Scarpetta

† These cautions are to deter the reader from naively expanding (1.1) as a power series in λ_0.

(1974a, b). This is the content of §§ 3 and 4. In particular we are interested in understanding the loss of information between asking for exact solutions and asking only for perturbation series.

Thirdly (§ 5) we examine the need to supplement the branching equations with renormalisation-group-type equations to extract as much information as possible from the formal path integrals.

Finally, we consider the alternative augmented formalism (Klauder 1977) for a theory with scale-*invariant* measure. As indicated in (Klauder 1981a, b), for non-canonical quantisation it is sufficient that the new measure be scale *covariant*, as in (1.2). However, the IVM that we have already mentioned (Klauder 1975, 1979b) has a scale-*invariant* measure

$$\mathscr{D}'[\Lambda\varphi]=\mathscr{D}[\varphi] \qquad \Lambda(x)>0, \forall x \tag{1.7}$$

and it is likely that such measures will be particularly important, deserving special study. The augmented formalism proposed by Klauder (1977) for a scale-*invariant* measure is to re-express $Z'[h]$ of (1.1) as

$$Z'[h,j]=\int \mathscr{D}[\varphi]\mathscr{D}[\chi]\exp -\int \mathrm{d}x(-\mathrm{i}h\varphi-\mathrm{i}j\chi+\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2+\lambda_0\varphi^4+\tfrac{1}{2}\eta_0\varphi^2\chi^2) \tag{1.8}$$

where η_0 is an arbitrary constant, and $\mathscr{D}[\varphi]$, $\mathscr{D}[\chi]$ are canonical translation-invariant measures normalised so that $Z'[0,0]=1$.

A further claim of Nouri-Moghadam and Yoshimura (1978) is that (iii) the augmented formalism contains no more information than the scale-covariant formalism (1.1), even though it enables us to obtain point-split variants of the equations (1.6). We examine this claim in § 6.

We stress that this paper is concerned with the relatively limited exercise of identifying those Green functions, or parts of Green functions, that are not constrained by the theory. As well as considering the problems mentioned above, Nouri-Moghadam and Yoshimura (1978) attempted the much more complicated problem of determining the Green functions in terms of the unconstrained components. We have nothing to say about this problem.

2. Discussion

Before examining the subtracted scale-covariant branching equations (1.6), let us recapitulate the main results of the translation-covariant Schwinger–Dyson equations that follow from the generating functional

$$Z[h]=\int \mathscr{D}[\varphi]\exp -\int \mathrm{d}x(-\mathrm{i}h\varphi+\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2+\lambda_0\varphi^4). \tag{2.1}$$

This Z satisfies the formal functional differential equation

$$\left\{h(x)-K_x\frac{\delta}{\delta h(x)}-4\lambda_0\frac{\delta^3}{\delta h(x)^3}\right\}Z[h]=0. \tag{2.2}$$

The branching equations for the *unconnected* Green functions

$$\mathrm{i}^n G_n(x_1\ldots x_n)=\left.\frac{\delta^n Z[h]}{\delta h(x_1)\ldots\delta h(x_n)}\right|_{h=0} \tag{2.3}$$

that follow from (2.2) are

$$\delta(x-x_1)-K_xG_2(xx_1)-4\lambda_0G_4(xxxx_1)=0 \qquad (2.4a)$$

$$[\delta(x-x_1)G_2(x_2x_3)+\delta(x-x_2)G_2(x_1x_3)+\delta(x-x_3)G_2(x_1x_2)]$$
$$-K_xG_4(xx_1x_2x_3)-4\lambda_0G_6(xxxx_1x_2x_3)=0 \qquad (2.4b)$$

$$\sum_{r=1}^{2m+1}\delta(x-x_r)G_{2m}(x_1\ldots\hat{x}_r\ldots x_{2m+1})-K_xG_{2m+2}(xx_1\ldots x_{2m+1})$$
$$-4\lambda_0G_{2m+4}(xxxx_1\ldots x_{2m+1})=0 \qquad m>1. \qquad (2.4c)$$

These equations are surprisingly degenerate, in that they permit $G_2(x_1x_2)$ to be *arbitrarily* specified. However, it is well known that the Schwinger–Dyson equations embody the unambiguous Feynman diagram expansion.

To see how this happens we express each unconnected Green function G_n as a power series in λ_0

$$G_n=\sum G_n^{(p)}\lambda_0^p. \qquad (2.5)$$

It then follows that (i) because of the inhomogeneous δ function in (2.4a) all the $G_n^{(0)}$, $n\geq 2$ are *uniquely* determined; (ii) if we express the relationship between the $G^{(p)}$ and the $G^{(p+1)}$ formally as

$$LG^{(p+1)}=G^{(p)} \qquad (2.6)$$

the operator L is *uniquely* invertible.

The seeming paradox that the perturbation series for G_2 obtained from (2.4) is *unique* (order by order), whereas G_2 itself is arbitrary in (2.4), is resolved in two ways. Firstly, the requirement that the perturbation series exists (i.e. that for non-exceptional coordinates the regularised G_{2n} satisfy $\lim_{\lambda_0\to 0}|G_{2n}|<\infty$) can be interpreted as a boundary condition. Secondly, the perturbation series in λ_0 is asymptotic, and hence does not determine G_2 uniquely unless additional information is given beyond that expressed in the branching equation (2.4).

This means that if we are *only* interested in developing the λ_0-perturbation series for the Green functions G_{2n}, we can trade the boundary condition implicit in perturbation expansions and the ambiguity in resummation for the boundary condition G_2.

What is the situation for the scale-covariant equations (1.6), which we write out in greater detail as

$$[\delta(x-x_1)+\delta(x-x_2)]W_2(x_1x_2)-\lim_{x'\to x}K_xW_4(xx'x_1x_2)-4\lambda_0W_6(xxxxx_1x_2)=0 \qquad (2.7a)$$

$$[\delta(x-x_1)+\delta(x-x_2)+\delta(x-x_3)+\delta(x-x_4)]W_4(x_1x_2x_3x_4)$$
$$-\lim_{x'\to x}K_xW_6(xx'x_1x_2x_3x_4)-4\lambda_0W_8(xxxxx_1x_2x_3x_4)=0 \qquad (2.7b)$$

$$\left(\sum_{r=1}^{2m}\delta(x-x_r)\right)W_{2m}(x_1\ldots x_{2m})-\lim_{x\to x'}K_xW_{2m+2}(xx'x_1\ldots x_{2m})$$
$$-4\lambda_0W_{2m+4}(xxxxx_1\ldots x_{2m})=0 \qquad m>2. \qquad (2.7c)$$

A diagrammatic representation of these equations is given in I.

The most noticeable properties of these equations are that (a) there is no inhomogeneous term to set a scale for the W_{2m}; (b) the first equation contains

W_2, W_4, W_6. Although $W_0 = 0$ is a necessary boundary condition, no equation contains W_0 because of the subtraction procedure (1.5).

The immediate observation made by Nouri-Moghadam and Yoshimura (1978) is that equations (2.7) are so degenerate that both W_2 *and* W_4 can be, and need to be, *independently* specified before W_{2p} $(p > 2)$ can be calculated. They are therefore more degenerate than the Schwinger–Dyson equations (2.4).

However, we have seen that this latter degeneracy is not present in the λ-perturbation series for the SD equations. If we are only interested in λ-perturbation series, to what extent can we trade the implicit boundary conditions and the ambiguity of resummation for knowledge of W_2 and W_4 in this case?

We expand W_{2n} as the pseudo-perturbation series

$$W_{2n} = \sum_{p\geq 0} \lambda_0^p W_{2n}^{(p)}. \tag{2.8}$$

The zeroth-order equations are

$$[\delta(x-x_1)+\delta(x-x_2)]W_2^{(0)}(x_1x_2) - \lim_{x'\to x} K_x W_4^{(0)}(xx'x_1x_2) = 0$$

$$\begin{aligned}&[\delta(x-x_1)+\delta(x-x_2)+\delta(x-x_3)+\delta(x-x_4)]W_4^{(0)}(x_1x_2x_3x_4)\\ &\qquad - \lim_{x'\to x} K_x W_6^{(0)}(x'xx_1x_2x_3x_4) = 0\end{aligned} \tag{2.9a}$$

etc. We see that these equations cannot permit the absolute determination of the $W_{2n}^{(0)}$, but once $W_2^{(0)}$ is given, the remaining $W_{2n}^{(0)}$ are *not* arbitrary.

However, any hopes that the $W_{2n}^{(p)}$ are determined once $W_2^{(0)}$ is given are not sustained on examining the non-leading equations

$$\begin{aligned}&[\delta(x-x_1)+\delta(x-x_2)]W_2^{(p+1)}(x_1x_2) - \lim_{x'\to x} K_x W_4^{(p+1)}(xx'x_1x_2)\\ &\qquad = 4W_6^{(p)}(xxxxx_1x_2) \qquad p\geq 0\end{aligned} \tag{2.10a}$$

$$\begin{aligned}&[\delta(x-x_1)+\delta(x-x_2)+\delta(x-x_3)+\delta(x-x_4)]W_4^{(p+1)}(x_1x_2x_3x_4)\\ &\qquad - \lim_{x'\to x} K_x W_6^{(p+1)}(xx'x_1x_2x_3x_4)\\ &\qquad = 4W_8^{(p)}(xxxxx_1x_2x_3x_4)\end{aligned} \tag{2.10b}$$

$$\begin{aligned}&\left[\sum_{r=1}^{2m} \delta(x-x_i)\right] W_{2m}^{(p+1)}(x_1x_2\ldots x_{2m}) - \lim_{x'\to x} K_x W_{2m+2}^{(p+1)}(xx'x_1\ldots x_{2m})\\ &\qquad = 4W_{2m+4}(xxxxx_1\ldots x_{2m}).\end{aligned} \tag{2.10c}$$

As was observed in I, knowledge of $W_2^{(0)}$ (and hence $W_{2m}^{(0)}$) does not obviously enable us to determine $W_{2m}^{(1)}$ and hence higher-order terms. That is, writing (2.10) symbolically as

$$LW^{(p+1)} = W^{(p)} \tag{2.11}$$

L is *not* uniquely invertible in this case. Moreover, unlike the SD equations, it is difficult to see exactly what we gain in trade for the ambiguity of resummation and implicit boundary conditions in just requiring the pseudo-perturbation series.

Let us therefore try to distinguish the combinatoric problems (i.e. which W can be specified independently) from the 'ascending problems' of actually determining the dependent W from the independent ones†. These latter problems require interpretation of the singularities of the W at coincident points (i.e. the 'renormalisation' prescription).

To understand the combinatoric problem it is sufficient to consider simplified versions of the canonical and scale covariant theories in which the kinetic term K_x is replaced by m_0^2 in (2.4) and (2.7). This guarantees that the distributions W_{2m}, G_{2m} are only products of δ functions and makes the 'ascending' problems trivially solvable.

In the next section we shall see what this means for the canonical theory.

3. The static ultra-local model

Let us examine what happens when we drop the kinetic term in the canonical theory of (2.4). The functional differential equation satisfied by the unconnected Green function generating functional $Z_0[h]$ becomes, on replacing K_x by m_0^2 in (2.2),

$$\left\{h(x)+m_0^2\frac{\delta}{\delta h(x)}-4\lambda_0\frac{\delta^3}{\delta h(x)^3}\right\}Z_0[h]=0. \tag{3.1}$$

For the static ultra-local model (SULM) (Klauder 1975) to be non-trivial it must be regularised, either by putting it on a lattice, lattice size M^{-1} or by demanding that $\delta(0)=M^d$ (in d dimensions). The factor M *cannot* be eliminated from the theory.

Given this regularisation, the general solution to (3.1) is seen to be

$$Z_0[h]=\exp\int \mathrm{d}^d x\, M^d \ln z_0(h(x)) \tag{3.2}$$

where the function $z_0(h)$ for constant argument h satisfies the differential equation

$$\left(h+m^2\frac{\partial}{\partial h}-4\lambda\frac{\partial^3}{\partial h^3}\right)z_0(h)=0 \tag{3.3}$$

in which

$$m^2=m_0^2M^{-d} \qquad \lambda=\lambda_0M^{-d}. \tag{3.4}$$

The connected Green functions

$$\mathrm{i}^n W_n(x_1\ldots x_n)=\frac{\delta^n}{\delta h(x_1)\ldots\delta h(x_n)}\ln Z_0[h]\bigg|_{h=0} \tag{3.5}$$

are obtained from z_0 in the following way.

If w_n is defined by

$$\mathrm{i}^n w_n\equiv\frac{\mathrm{d}^n}{\mathrm{d}h^n}\ln z_0(h)\bigg|_{h=0} \tag{3.6}$$

it follows that

$$W_n(x_1\ldots x_n)=w_n\prod_{r=2}^{n}\delta(x_1-x_r). \tag{3.7}$$

† This can be a serious problem even for the canonical theory (Yoshimura 1979).

To establish a correspondence with the previous section we examine the linear relationships satisfied by the τ_n, defined by

$$\mathrm{i}^n\tau_n \equiv \frac{\mathrm{d}^n}{\mathrm{d}h^n} z_0(h)\bigg|_{h=0}. \tag{3.8}$$

Knowing the τ_n we can construct the w_n, hence the W_n, and finally the G_n for the model.

We assume that the τ_{2p+1} are zero for all non-negative integer p. If τ is the column vector with pth element

$$(\tau)_p = \tau_{2p} \tag{3.9}$$

the recurrence relations are most usefully expressed as the matrix equation

$$(-m^2\mathbb{1} + L - 4\lambda D)\tau = -e_1 \tag{3.10}$$

where L is the *lower* semi-matrix

$$L_{ij} = (2j+1)\delta_{i,j+1} \tag{3.11}$$

D the *upper* semi-matrix

$$D_{ij} = \delta_{i+1,j} \tag{3.12}$$

and e_1 the column matrix

$$(e_1)_p = \delta_{1,p}. \tag{3.13}$$

The matrix $(-m^2\mathbb{1} + L - 4\lambda D)$ does *not* have a unique right-hand reciprocal $(-m^2\mathbb{1} + L - 4\lambda D)^{-1}$ satisfying

$$(-m^2\mathbb{1} + L - 4\lambda D)(-m^2\mathbb{1} + L - 4\lambda D)^{-1} = 1 \tag{3.14}$$

because of the presence of the upper semi-matrix. Thus, equation (3.10) cannot be solved uniquely, permitting a family of solutions labelled by τ_2.

However, since $(-m^2\mathbb{1} + L)$ is a *lower* semi-matrix it does have a *unique* right-hand reciprocal $(-m^2\mathbb{1} + L)^{-1}$. Thus (3.10) can be *uniquely* re-expressed as

$$\tau = -(-m^2\mathbb{1} + L)^{-1}e_1 + 4\lambda(-m^2\mathbb{1} + L)^{-1}D\tau. \tag{3.15}$$

If we expand τ as a power series in λ as

$$\tau = \sum_{p\geq 0} \tau^{(p)}\lambda^p \tag{3.16}$$

we therefore have a *unique* perturbation series solution†

$$\tau^{(p+1)} = 4(-m^2\mathbb{1} + L)^{-1}D\tau^{(p)} \tag{3.17}$$

with

$$\tau^{(0)} = -(-m^2\mathbb{1} + L)^{-1}e_1. \tag{3.18}$$

This example demonstrates very simply how degenerate equations permitting arbitrary $\tau_2 = w_2$ and hence G_2 have unique perturbation series expansions.

† This corresponds to expressing $(-m^2\mathbb{1} + L - 4\lambda D)^{-1}$ as $[\mathbb{1} - 4\lambda(-m^2\mathbb{1} + L)^{-1}D]^{-1}(-m^2\mathbb{1} + L)^{-1} = [\Sigma_{p\geq 0}[4\lambda(-m^2\mathbb{1} + L)^{-1}D]^p(-m^2\mathbb{1} + L)^{-1}$ which can be seen *not* to converge for any $\lambda \neq 0$.

It is informative to see how we would have reached the same conclusion directly from equation (3.3). Imposing the *two* boundary conditions

(i) $$z_0(0)=1 \tag{3.19a}$$

(ii) $$z_0(h)=z_0(-h) \tag{3.19b}$$

z_0 has the form

$$z_0(h)=\frac{Z(m^2,\lambda,h)+\gamma(m^2,\lambda)Z(-m^2,\lambda,\mathrm{i}h)}{Z(m^2,\lambda,0)+\gamma(m^2,\lambda)Z(-m^2,\lambda,0)} \tag{3.20}$$

where

$$Z(m^2,\lambda,h)=\int_{-\infty}^{\infty}\mathrm{d}u\,\exp(\mathrm{i}hu-\tfrac{1}{2}m^2u^2-\lambda u^4) \tag{3.21}$$

and $\gamma(m^2,\lambda)$ is an *arbitrary* function of m^2, λ.

As before, all Green functions are fixed once τ_2 (or equivalently G_2) is specified. Each choice of τ_2 corresponds to a choice of $\gamma(m^2,\lambda)$.

From (3.20) we see that, in an obvious notation,

$$\tau_{2p}=\frac{\alpha^+_{2p}+\gamma\alpha^-_{2p}}{\alpha^+_0+\gamma\alpha^-_0} \tag{3.22}$$

where

$$\alpha^{\pm}_{2p}=(-1)^{p/2\mp p/2}\left(\frac{1}{32\lambda}\right)^{p/2}\frac{(2p)!}{(p)!}U(p,\pm(m^4/8\lambda)^{1/2}). \tag{3.23}$$

The $U(a,x)$ are the usual parabolic cylinder functions.

For small λ

$$\alpha^-_{2p}/\alpha^+_{2p}=\mathrm{O}(\lambda^{-p}\exp(m^4/16\lambda)). \tag{3.24}$$

If we now restrict ourselves merely to considering the perturbation series for τ_{2p}, we require that

$$\lim_{\lambda\to 0^+}|\tau_{2p}(m^2,\lambda)|<\infty. \tag{3.25}$$

This can only be satisfied if

$$\gamma=0 \tag{3.26}$$

or

$$\gamma=\mathrm{O}(\exp(-am^4/16\lambda)) \qquad a>1. \tag{3.27}$$

This is equivalent to requiring that $\gamma\alpha^-_{2p}$ has *zero* asymptotic series.

This is, the condition (3.25) that the perturbation series exists guarantees that it is *unique* (but not uniquely summable) as we had already seen in the recurrence relations.

4. The independent-value model

We now drop the kinetic term in the scale-covariant theory of (1.3). The functional differential equation for the generating functional $Z'_0[h]$ becomes

$$\left\{h(x)\frac{\delta}{\delta h(x)}+m_0^2:\frac{\delta^2}{\delta h(x)^2}:-4\lambda_0:\frac{\delta^4}{\delta h(x)^4}:\right\}Z'_0[h]=0. \tag{4.1}$$

The general solution to this independent-value model (IVM) is of the form (Klauder 1979b)

$$Z'_0[h]=\exp\int \mathrm{d}x\, \mathscr{W}(h(x)) \tag{4.2}$$

where $\mathscr{W}_0$ satisfies the *linear* subtracted equation

$$\frac{h\,\partial \mathscr{W}_0}{\partial h}+\bar{m}^2\left(\frac{\partial^2 \mathscr{W}_0}{\partial h^2}-\frac{\partial^2 \mathscr{W}_0}{\partial h^2}\bigg|_{h=0}\right)-4\bar{\lambda}\left(\frac{\partial^4 \mathscr{W}_0}{\partial h^4}-\frac{\partial^4 \mathscr{W}_0}{\partial h^4}\bigg|_{h=0}\right)=0 \tag{4.3}$$

with

$$\bar{m}^2=\delta(0)m_0^2 \qquad \bar{\lambda}=\delta(0)^3\lambda_0. \tag{4.4}$$

In d space–time dimensions, we introduce renormalised mass m and coupling constant λ by

$$bm^2=\bar{m}^2 \qquad b^3\lambda=\bar{\lambda} \tag{4.5}$$

where b, of dimension $(\text{mass})^d$, plays a role analogous to M^d in equation (3.4).

The connected Green functions

$$\mathrm{i}^n W'_n(x_1\ldots x_n)=\frac{\delta^n}{\delta h(x_1)\ldots\delta h(x_n)}\ln Z'_0[h]\bigg|_{h=0} \tag{4.6}$$

are obtained from $\mathscr{W}_0(h)$ as

$$W'_n(x_1\ldots x_n)=w'_n\prod_{r=2}^{n}\delta(x_1-x_r) \tag{4.7}$$

where

$$\mathrm{i}^n w'_n=\frac{\mathrm{d}^n}{\mathrm{d}h^n}\mathscr{W}_0(h)\bigg|_{h=0}. \tag{4.8}$$

As in the previous section we first examine the branching equations implied by (4.3). If (taking $w'_{2p+1}=0$, all non-negative integer p) w' is the column-vector

$$(w')_p=w'_{2p} \qquad p\geqslant 1 \tag{4.9}$$

the branching equations have the matrix form

$$(2\Lambda-\bar{m}^2D-4\bar{\lambda}D^2)w'=0 \tag{4.10}$$

where Λ is the diagonal matrix

$$\Lambda_{ij}=j\delta_{ij} \tag{4.11}$$

and D is given in (3.12).

By definition, the homogeneous equation (4.10) cannot have a unique solution. Let us therefore separate off w_2', to rewrite (4.10) as

$$(-\bar{m}^2\mathbb{1}+L'-4\bar{\lambda}D)\bar{w}=-2w_2'e_1 \tag{4.12}$$

where $\bar{w}$ is the column vector

$$\bar{w}=Dw' \qquad (\bar{w})_p=w'_{p+2} \tag{4.13}$$

and L' is the *lower* semi-matrix

$$L'_{ij}=2(j+1)\delta_{i,j+1}. \tag{4.14}$$

Equation (4.14) can be compared directly with (3.10). As in that case, the matrix $(-\bar{m}^2\mathbb{1}+L'+4\bar{\lambda}D)$ does *not* have a unique right-hand reciprocal. In consequence, (4.12) cannot be uniquely solved for $\bar{w}$ once w_2' is given, permitting a family of solutions labelled by w_4' (in addition to w_2').

However, since $(-\bar{m}^2\mathbb{1}+L')$ does have a *unique* right-hand reciprocal we can write (4.12) uniquely as

$$\bar{w}=-2w_2'(-\bar{m}^2\mathbb{1}+L')^{-1}e_1+4\bar{\lambda}(-\bar{m}^2\mathbb{1}+L')^{-1}Dw. \tag{4.15}$$

If we expand $w_2'^{-1}\bar{w}$ as a power series in $\bar{\lambda}$ as

$$w_2'^{-1}\bar{w}=\sum_{p\geqslant 0}\bar{\lambda}^p\tilde{w}^{(p)} \tag{4.16}$$

we see that $\tilde{w}^{(p)}$ is uniquely determined from

$$\tilde{w}^{(p+1)}=4(-\bar{m}^2\mathbb{1}+L')^{-1}D\tilde{w}^{(p)} \qquad p\geqslant 1 \tag{4.17a}$$

with

$$\tilde{w}^{(0)}=-2(\bar{m}^2\mathbb{1}+L')e_1. \tag{4.17b}$$

That is, once the perturbation series for w_2 is given, all other perturbation series are uniquely determined.

To see how we would have obtained this result directly from (4.3) we impose the *three* boundary conditions

$$\begin{aligned}&\text{(i)} && \mathcal{W}_0(0)=0\\ &\text{(ii)} && \mathcal{W}_0(h)=\mathcal{W}_0(-h)\end{aligned} \tag{4.18}$$

implying $w_0=w_1=w_3=0$†. This gives $\mathcal{W}_0(h)$ to have the form

$$\mathcal{W}_0(h)=a(\bar{m}^2,\bar{\lambda})w'(\bar{m}^2,\bar{\lambda},h)+c(\bar{m}^2,\bar{\lambda})w'(-\bar{m}^2,\bar{\lambda},\mathrm{i}h) \tag{4.19}$$

where

$$w'(\bar{m}^2,\bar{\lambda},h)=\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{|u|}(1-\cos uh)\exp-(\tfrac{1}{2}\bar{m}^2u^2+\bar{\lambda}u^4). \tag{4.20}$$

The specification of the arbitrary functions $a(\bar{m}^2,\bar{\lambda})$, $c(\bar{m}^2,\bar{\lambda})$ is equivalent to independently specifying w_2' and w_4'.

Writing w_{2p}' in an obvious notation as

$$w_{2p}'=aw_{2p}^{+}+cw_{2p}^{-} \tag{4.21}$$

† The solutions to (4.3) need *five* boundary conditions to be uniquely determined.

we have

$$w_{2p}^{-}/w_{2p}^{+} = \mathrm{O}(\bar{\lambda}^{-p}\exp(\bar{m}^4/16\bar{\lambda})) \tag{4.22}$$

as in (3.24).

On restricting ourselves just to the perturbation series we require that, in order that it exists,

$$\lim_{\bar{\lambda}\to 0^-} |w'_{2p}(\bar{m}^2, \bar{\lambda})| < \infty. \tag{4.23}$$

This forces cw_{2p}^{-} to have *zero* asymptotic series for all p. On assuming this, we see that the *asymptotic series in* $\bar{\lambda}$ *for* $w_{2p}w_2^{-1}$ (independent of a) are now uniquely determined even though they are *not* uniquely summable to $w_{2p}w_2^{-1}$.

5. Renormalisation group equations

Our analysis so far has shown the extent to which the scale-covariant equations (4.1) carry less information than the translation-covariant equations (3.1).

However, this difference in information is not a true reflection of our lack of knowledge about the solutions to the respective models. For example, a lattice calculation for the SULM shows (Kainz 1975) that it is completely determined, rather than just its perturbation series. Similarly, a Fock space calculation for the IVM (Klauder 1975) shows that is is completely determined up to a *single* overall scale parameter.

The additional information that we have not yet used comprises the branching equations of the 'second kind' (in the terminology of Caianiello and Scarpetta 1974a, b). For *both* the SULM and the IVM they are

$$\frac{\partial}{\partial(\frac{1}{2}m_0^2)} Z[h] = \int \mathrm{d}x : \frac{\delta^2}{\delta h(x)^2} : Z[h] \tag{5.1}$$

$$\frac{\partial}{\partial\lambda_0} Z[h] = \int \mathrm{d}x : \frac{\delta^4}{\delta h(x)^4} : Z[h] \tag{5.2}$$

(where Z denotes Z_0 or Z'_0) reflecting the change in Z as m_0^2 and λ_0 are changed. These equations follow from the Schwinger action principle and are automatically satisfied by the formal path integrals. They have been examined for the two models in question in a different context (Marinaro 1976) and we shall be brief.

It is sufficient for our purpose to see the constraints they impose on $z_0(h)$ and $\mathcal{W}_0(h)$.

Let us first consider the SULM. Equations (5.1) become

$$\frac{\partial}{\partial(\frac{1}{2}m^2)} z_0(h) = \frac{\partial^2}{\partial h^2} z_0(h) - \frac{\partial^2}{\partial h^2} z_0(h)\bigg|_{h=0} z_0(h) \tag{5.3}$$

and

$$\frac{\partial}{\partial\lambda} z_0(h) = \frac{\partial^4}{\partial h^4} z_0(h) - \frac{\partial^4}{\partial h^4} z_0(h)\bigg|_{h=0} z_0(h). \tag{5.4}$$

It is possible to factorise $z_0(h)$ as

$$z_0(h) = E(h)/E(0) \tag{5.5}$$

such that (5.3) and (5.4) become

$$\frac{\partial}{\partial(\frac{1}{2}m^2)}E(h)=\frac{\partial^2}{\partial h^2}E(h) \tag{5.6}$$

and

$$\frac{\partial}{\partial\lambda}E(h)=\frac{\partial^4}{\partial h^4}E(h). \tag{5.7}$$

If the derivatives of E are defined by

$$\mathrm{i}^n\bar{\tau}_n=\frac{\partial^n}{\partial h^n}E(h)\bigg|_{h=0} \tag{5.8}$$

it follows that

$$\tau_n=\bar{\tau}_n/\bar{\tau}_0. \tag{5.9}$$

Equations (5.6) and (5.7) then imply

$$\frac{\partial}{\partial(\frac{1}{2}m^2)}\bar{\tau}_{2p}=-\bar{\tau}_{2p+2} \qquad p\geqslant 0 \tag{5.10}$$

$$\frac{\partial}{\partial\lambda}\bar{\tau}_{2p}=\bar{\tau}_{2p+4} \qquad p\geqslant 0. \tag{5.11}$$

In the notation of (3.22)

$$\bar{\tau}_{2p}=\alpha^+_{2p}+\gamma\alpha^-_{2p}. \tag{5.12}$$

Since $\alpha^\pm_{2p}$ individually satisfy both (5.10) and (5.11), as follows directly from (3.21), these equations become

$$\frac{\partial\gamma}{\partial(\frac{1}{2}m^2)}=\frac{\partial\gamma}{\partial\lambda}=0 \tag{5.13}$$

with solution

$$\gamma=\text{constant}. \tag{5.14}$$

That is, whereas the SD branching equations gave a family of solutions depending on the arbitrary *function* $\tau_2(m^2,\lambda)$, the further imposition of (5.1) and (5.2) produces a family of solutions depending on a single parameter γ.

Furthermore, if we now require from (3.25) that the perturbation series exists, only the solution $\gamma=0$ of (3.26) is tenable. That is, the perturbation series is both *unique* and *uniquely* resummable†.

This is difficult to see in the matrix formulation. If

$$(\tau')_p=\bar{\tau}_{2p-2} \qquad p\geqslant 1 \tag{5.15}$$

and

$$\bar{\tau}=D\tau'$$

† We do not expect this to be true for the full canonical theory after renormalisation has been implemented (because of the possibility of renormalons (Lautrup 1977, Parisi 1978), for example).

equations (3.10), (5.1) and (5.2) become

$$(-m^2\mathbb{1}+L-4\lambda D)\bar{\tau}=-e_1\bar{\tau}_0 \tag{5.16}$$

$$\frac{\partial}{\partial(\frac{1}{2}m^2)}\tau'=-D\tau' \tag{5.17}$$

$$\frac{\partial}{\partial\lambda}\tau'=-D^2\tau'. \tag{5.18}$$

The intersection of (5.17) and (5.18) with the non-uniquely solvable (5.16) is obscure†.

Let us now consider the IVM of the previous section for which equations (5.1) and (5.2) become

$$\frac{\partial}{\partial(\frac{1}{2}\bar{m}^2)}\mathscr{W}_0(h)=-\frac{\partial^2}{\partial h^2}\mathscr{W}_0(h) \tag{5.19}$$

and

$$\frac{\partial}{\partial\bar{\lambda}}\mathscr{W}_0(h)=\frac{\partial^4}{\partial h^4}\mathscr{W}_0(h). \tag{5.20}$$

Imposing these conditions on the exact solution (4.19) we see from (4.20) that $w'(\bar{m}^2,\bar{\lambda},h)$ and $w'(-\bar{m}^2,\bar{\lambda},ih)$ individually satisfy (5.19) and (5.20).

Equations (5.19) and (5.20) therefore imply

$$\frac{\partial a}{\partial(\frac{1}{2}\bar{m}^2)}=\frac{\partial c}{\partial(\frac{1}{2}\bar{m}^2)}=0 \tag{5.21}$$

and

$$\frac{\partial a}{\partial\bar{\lambda}}=\frac{\partial c}{\partial\bar{\lambda}}=0.$$

That is, both a and c are constant.

Again, the imposition of (5.1) and (5.2) reduces $\mathscr{W}_0$ from a family of solutions dependent on two functions to a family dependent on two constant parameters.

If we now impose (4.23), so that the perturbation series exists, we see from (4.22) that we must have $c=0$ to give

$$\mathscr{W}_0(h)=a\int\frac{\mathrm{d}u}{|u|}(1-\cos uh)\exp-(\tfrac{1}{2}\bar{m}^2u^2+\bar{\lambda}u^4). \tag{5.22}$$

That is, whereas the imposition of (5.1) and (5.2) gives no further information on the form of the λ-perturbation series for the ULM, this is not the case for the IVM, by virtue of the restrictions on a. Up to this constant scale factor the asymptotic series in λ, and its resummation, is *unique*‡.

In terms of the matrix equations, equations (4.12), (5.19) and (5.20) become

$$(-\bar{m}^2\mathbb{1}+L'-4\bar{\lambda}D)\bar{w}=-2w_2'e_1 \tag{5.23}$$

$$\frac{\partial}{\partial(\frac{1}{2}\bar{m}^2)}w'=Dw' \tag{5.24}$$

† We note from the above that the perturbation series (3.16)–(3.18) automatically satisfies (5.1) and (5.2).

‡ We would again expect this unique resummability to be an artifice of the restricted model, and not shared by the full theory.

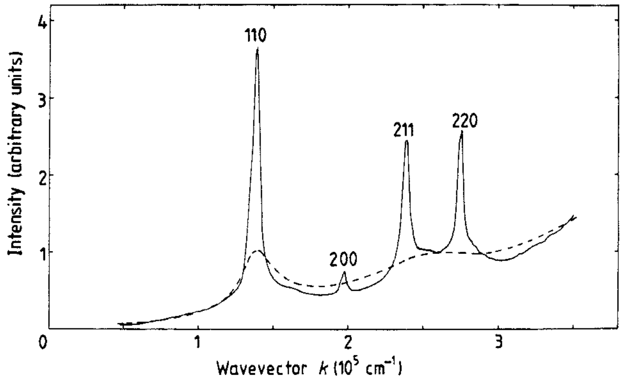

Figure 1. Static light scattering from a colloidal solution of latex spheres with diameter 0.091 μm at nominal concentration of 5×10^{12} particles/cm^3 in the liquid (broken line) and crystalline (full line) phase. No corrections for multiple scattering and turbidity effects are made. The numbers are the indices of the Bragg peaks according to a BCC structure.

4. Results and discussion

4.1. Colloidal liquids

Most of the experimental data on the colloidal liquids has been published in earlier papers (Grüner and Lehmann 1979, 1980a). The purpose of this section is to review the data according to the theory in § 2 and to make a direct comparison with the data on colloidal crystals possible.

All measured intensity correlation functions were corrected for multiple scattering (Grüner and Lehmann 1980b), and it turned out that two exponentials were always sufficient to provide an excellent fit to the data. The two-exponential form of the functions is an experimental justification for the visoelastic approximation made in § 2. The results for one particular concentration are shown in figure 2 together with the static structure factor $S(\boldsymbol{k})$ from static light scattering measurements. It can be seen that the long-time decay constant goes towards the short-time decay for small $\boldsymbol{k}$ vectors.

Since the samples are rather monodisperse, they do not show the incoherent self-diffusion term (Pusey 1980, Weisman 1980) present in previous experiments by Pusey (1978) and Dalberg *et al* (1978). From the data of the fit we are able to extract the memory function, which is, apart from numerical constants, the internal longitudinal viscosity. From equations (8), (6) and (5) it follows in the limit of $\omega\to0$ that

$$\tilde{M}_{\rm red}(\boldsymbol{k},\omega=0)=\frac{k^2}{m\rho\omega_{\rm S}}[\zeta(\boldsymbol{k},\omega=0)+\tfrac{4}{3}\eta(\boldsymbol{k},\omega=0)]=\frac{\Gamma_1+\Gamma_2-\Gamma_{\rm C}-\Gamma_{\rm M}}{\Gamma_{\rm M}}. \qquad (17)$$

$\Gamma_{\rm C}$ is the first cumulant given by equation (9) and $\Gamma_{\rm M}$ is the decay constant of the memory function (equation (3)), which is given by the equation

$$\Gamma_{\rm M}=\Gamma_1\Gamma_2/\Gamma_{\rm C}. \qquad (18)$$

With equation (17) we have defined a reduced, dimensionless quantity $\tilde{M}_{\rm red}$. In our earlier papers (Grüner and Lehmann 1979, 1980a) we have analysed our data in terms of a memory function formalism based on the Smoluchowsky equation. The and (5.2) gives

$$\left\{h(x)\frac{\delta}{\delta h(x)}-m_0^2\frac{\partial}{\partial(\frac{1}{2}m_0^2)}-4\lambda_0\frac{\partial}{\partial\lambda_0}\right\}Z_0[h]$$
$$=-\{\delta(0)-m_0^2G_2(xx)-4\lambda_0G_4(xxxx)\}Z_0[h]. \qquad (5.29)$$

By virtue of (2.4*a*) we know that (5.29) is satisfied by each side being identically zero. However, the left-hand side of (5.29) vanishes just on grounds of engineering dimensions. In consequence, the equation

$$\delta(0)-m_0^2G_2(xx)-4\lambda_0G_4(xxxx)=0 \qquad (5.30)$$

is, in some sense, a consequence of the renormalisation-group-type branching equations of the second kind.

That is, if instead of equations (2.4) (with K_x replaced by m_0^2) we only had equations (2.4*b*) and (2.4*c*), equation (2.4*a*) could have been reconstructed in the form (5.30). Thus, for the ULM we can essentially swap the 'second kind' branching equations for the dynamical equation (5.30). It is the absence of an equation like (2.4*a*) that causes so much difficulty in scale-covariant models.

Unfortunately, it is seen that a similar situation does not hold for the more relevant scale-covariant IVM. However, the IVM does have some peculiarities of its own and the comments above suggest that rather than attempt to formulate and impose renormalisation-group-type equations on a deficient set of branching equations (2.7), we should aim to supplement them with further dynamical equations that have the same content. The next section shows how this can be done.

6. The augmented formalism for scale-invariant measures

Let us restrict ourselves to the important case of scale-invariant measures.

Invoking the translation invariance of the measures $\mathscr{D}[\varphi]$, $\mathscr{D}[\chi]$ in (1.7) gives rise to the equations

$$\left(h(x)+K_x\frac{\delta}{\mathrm{i}\delta h(x)}-4\lambda_0\frac{\delta^3}{\mathrm{i}^3\delta h(x)^3}-\eta_0\frac{\delta^3}{\mathrm{i}^3\delta h(x)\delta j(x)^2}\right)Z'[h,j]=0 \qquad (6.1)$$

and

$$\left(j(x)-\eta_0\frac{\delta^3}{\mathrm{i}^3\delta h(x)^2\delta j(x)}\right)Z'[h,j]=0. \qquad (6.2)$$

Equation (6.2) is a constraint equation, whereas (6.1) is more dynamical. If the unconnected Green functions are defined by

$$\mathrm{i}^{m+n}G_{m,n}(x_1\ldots x_m;y_1\ldots y_n)=\left.\frac{\delta^{m+n}Z'[h,j]}{\delta h(x_1)\ldots\delta h(x_m)\delta j(y_1)\ldots\delta j(y_n)}\right|_{h=j=0} \qquad (6.3)$$

the branching equations (and their diagrammatic representations) have been given in I.

In Nouri-Moghadam and Yoshimura (1978) it was surprisingly argued that the branching equations following from (6.1) and (6.2) were *not* less degenerate than (2.7). This is patently not the case, as can be seen on differentiating (6.1) with respect to $h(y)$, differentiating (6.2) with respect to $j(y)$ and subtracting.

This gives

$$\left[h(x)\frac{\delta}{\mathrm{i}\delta h(y)}-j(x)\frac{\delta}{\mathrm{i}\delta j(y)}+\frac{\delta}{\mathrm{i}\delta h(y)}K_x\frac{\delta}{\mathrm{i}\delta h(x)}-4\lambda_0\frac{\delta^4}{\mathrm{i}^4\delta h(x)^3\delta h(y)}\right.$$
$$\left.-\eta_0\left(\frac{\delta^4}{\delta h(x)\delta h(y)\delta j(x)^2}-\frac{\delta^4}{\delta j(x)\delta j(y)\delta h(x)^2}\right)\right]Z'[h,j]=0 \qquad (6.4)$$

as a replacement for either (6.1) or (6.2).

On setting $j=0$ and $x=y$ (6.4) becomes

$$\left\{h(x)\frac{\delta}{\mathrm{i}\delta h(x)}+\frac{\delta}{\mathrm{i}\delta h(x)}K_x\frac{\delta}{\mathrm{i}\delta h(x)}-4\lambda_0\frac{\delta^4}{\delta h(x)^4}\right\}Z'[h]=0 \qquad (6.5)$$

identical to (1.4) *but* for the subtraction procedure.

That is, by using the augmented formalism we seem to gain the *additional* equation

$$\lim_{x'\to x} K_x W_2(xx')-4\lambda_0 W_4(xxxx)=0. \qquad (6.6)$$

We have suggested earlier that an additional equation was just what we needed as an alternative to the renormalisation-group-type equations of the second kind. In fact, with this equation it is at least plausible that, once $W_2^{(0)}$ is specified, all $W_{2n}^{(p)}$ are constrained, if not determined. This was the hope expressed, but not fulfilled, in § 2.

Should we take (6.6) seriously as *new* information that may save us from having to use renormalisation-group-like equations, or treat it as a defect in our presentation of the augmented formalism that should be eliminated? We shall treat this problem in some detail in the following paper (Ebbutt and Rivers 1982b) in which we argue that (6.6) should be accepted as a true formal equation for the full scale-covariant theory of (1.1) with scale-invariant measure.

This, in turn, will indicate the nature of the additional information required to supplement the branching equations for the more general case of theories with scale-covariant (rather than invariant) measures.

7. Conclusion

We have re-appraised the criticisms of the scale-covariant and augmented theories made in Nouri-Moghadam and Yoshimura (1978) (comments (i), (ii) and (iii) of the introduction). Our work suggests that these criticisms are unduly negative, and more positive conclusions can be drawn.

Firstly, although Klauder's subtracted scale-covariant branching equations (1.6) are very degenerate, we are reminded that the Schwinger–Dyson equations are also (although less) degenerate. We know that, if we are only interested in perturbation series, this latter degeneracy can be traded for the implicit boundary condition (in demanding a perturbative solution) and the ambiguity of resummation†. For Klauder's equations the degeneracy is also reduced but not eliminated, by requiring only the existence of a pseudo-perturbation series.

† Given the difficulties of resummation of canonical renormalisable theories, we consider the establishment of a perturbation series as a success in its own right.

In order to reduce the degeneracy further we have two options. Firstly, we can impose the additional constraint equations that are the analogues of the Callan–Symanzik and renormalisation group equations. We do not yet know how to do so, but we believe that, when we can, the degeneracy in the scale-covariant pseudo-perturbation theory will be no more than an undetermined scale factor.

Alternatively, we look for an additional dynamical equation that will contain the same information as these renormalisation-group-like equations. For the case of scale-invariant measures the augmented formalism seems to provide such an additional equation by preventing the subtraction procedure of (1.4). This equation will be much easier to implement than a renormalisation-group analysis that needs to be constructed *ab initio*. At the level of argument presented here, such depends on the extent to which we consider the IVM to be a special case.

We shall present additional, and very different, arguments for the validity of the naive augmented formalism in paper III.

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